ON TELGÁRSKY'S QUESTION CONCERNING β-FAVORABILITY OF THE STRONG CHOQUET GAME

LÁSZLÓ ZSILINSZKY

Abstract. Answering a question of Telgársky in the negative, it is shown that there is a space which is β-favorable in the strong Choquet game, but its nonempty W_δ-subspaces are of the 2nd category in themselves.

1. Introduction

One of the well-known applications of the *Banach-Mazur game* [HMC] (also known as *Choquet game* [Ke]) is a characterization of Baire topological spaces (i.e. spaces where nonempty open subspaces are of the 2nd category in themselves); namely, a space is Baire iff the first player in the Banach-Mazur game has no winning strategy [Ox, Kr]. The *strong Choquet game* [Ke] is a modification of the Banach-Mazur game that also yields nice characterizations of various completeness-type properties (see below). In particular, *Telgársky* [Te] noticed - somewhat analogously to the above Baire space characterization - that in any topological space, if the first player in the strong Choquet game has no winning strategy, then the nonempty W_δ-subspaces are of the 2nd category in themselves (where W_δ-sets are generalizations of G_δ-sets introduced by *Wicke* and *Worrell* [WW]), and asked whether it is actually a characterization. This indeed is the case, e.g., in 1st countable T_1-spaces [Zs]; however, we will show that a counterexample exists in the non-1st-countable case, and so Telgársky's conjecture fails.

First we introduce the relevant notions and terminology: let $\mathcal{B}$ be a base for a topological space X, and denote $\mathcal{E} = \{(x,U) \in X \times \mathcal{B} : x \in U\}$. In the *strong Choquet game* $Ch(X)$ players β and α alternate in choosing $(x_n, V_n) \in \mathcal{E}$ and $U_n \in \mathcal{B}$, respectively, with β choosing first, so that for each $n \in \omega$, $x_n \in U_n \subseteq V_n$, and $V_{n+1} \subseteq U_n$. The play $(x_0, V_0), U_0, \ldots, (x_n, V_n), U_n, \ldots$ is won by β, if $\bigcap_n V_n = \emptyset$; otherwise, α wins. A *strategy* in $Ch(X)$ for β is a function $\sigma : \mathcal{B}^{<\omega} \to \mathcal{E}$ such that $\sigma(\emptyset) = (x_0, V_0)$, and $\sigma(U_0, \ldots, U_{n-1}) = (x_n, V_n)$ with $V_n \subseteq U_{n-1}$ for all $(U_0, \ldots, U_{n-1}) \in \mathcal{B}^n$, $n \geq 1$. A strategy σ for

2010 *Mathematics Subject Classification.* Primary 91A44; Secondary 54E52, 54B20.
Key words and phrases. strong Choquet game, Vietoris topology, Baire category, W_δ-set, Tychonoff square.

β is a *winning strategy*, if β wins every run of $Ch(X)$ compatible with σ. We will say that $Ch(X)$ *is β-favorable*, provided β has a winning strategy in $Ch(X)$. Strategies for α in $Ch(X)$, and α-favorability of $Ch(X)$ can be defined analogously [Ke].

The strong Choquet game was introduced by *Choquet* in [Ch], who showed that in a metrizable space X, α has a winning strategy in $Ch(X)$ iff X is completely metrizable. Later, *Debs* [De] and *Telgársky* [Te] independently showed that if X is metrizable, then β has a winning strategy in $Ch(X)$ iff X is contains a closed copy of the rationals (i.e. iff X is not hereditarily Baire). The strong Choquet game has been studied in non-metrizable settings as well (cf. [Po],[GT],[Ma],[CP],[BLR],[DM],[Zs]).

Let $Y \subseteq X$. A *sieve* of Y (cf. [CCN], [Gr]) in X is a pair (G, T), where $(T, <)$ is a tree of height ω with levels $T_0, T_1, \dots$, and G is a function on T with X-open values such that

- $\{G(t) : t \in T_0\}$ is a cover of Y,
- $Y \cap G(t) = \bigcup\{Y \cap G(t') : t' \in T_{n+1}, t' > t\}$ for each n, and $t \in T_n$,
- $t \leq t' \Rightarrow G(t) \supseteq G(t')$ for each $t, t' \in T$.

We will say that Y is a *W_δ-set* in X, if Y has a sieve (G, T) in X such that $\bigcap_n G(t_n) \subseteq Y$ for each branch (t_n) of T. A G_δ-set is also a W_δ-set. A Tychonoff space is *sieve complete* iff it is a W_δ subspace of a compact space iff it is a continuous open image of a Čech-complete space [WW, Theorem 4]; in particular, sieve complete spaces are of the 2nd Baire category.

Denote by $CL(X)$ the set of all nonempty closed subsets of a T_1-space X, and for any $S \subseteq X$ put $S^- = \{A \in CL(X) : A \cap S \neq \emptyset\}$ and $S^+ = \{A \in CL(X) : A \subseteq S\}$. The *Vietoris topology* [Mi] τ_V on $CL(X)$ has subbase elements of the form U^- and U^+, where $\emptyset \neq U \subseteq X$ is open; so a base for τ_V is

$$\mathcal{B}_V = \{U^+ \cap \bigcap_{i \leq n} U_i^- : n \in \omega,\ U, U_i \subseteq X \text{ open}\}.$$

The space $(CL(X), \tau_V)$ is T_2 (resp. T_3) iff X is T_3 (resp. T_4), and $(CL(X), \tau_V)$ is compact iff X is compact [Mi]. If A is an open (resp. closed) subspace of X, then $CL(A)$ is an open (resp. closed) subspace of $CL(X)$; X embeds as a subspace in $CL(X)$ (it embeds as a closed subspace iff X is T_2). The following lemma will be used in the main result:

Lemma 1.1. [Mi, Lemma 2.3.1]
If $U^+ \cap \bigcap_{i \leq n} U_i^-,\ V^+ \cap \bigcap_{j \leq m} V_j^- \in \mathcal{B}_V$, then the following are equivalent:
(i) $U^+ \cap \bigcap_{i \leq n} U_i^- \subseteq V^+ \cap \bigcap_{j \leq m} V_j^-$
(ii) *$U \subseteq V$, and for every $j \leq m$ there is $i \leq n$ such that $U_i \subseteq V_j$.*

2. Main Result

The *Tychonoff square* is defined as $X = (\omega_1 + 1) \times (\omega_1 + 1) \setminus \{(\omega_1, \omega_1)\}$, where ω_1 is the first uncountable ordinal with the order topology.

Theorem 2.1. *If X is the Tychonoff square, then*
(i) *$Ch(CL(X))$ is β-favorable, and*
(ii) *every nonempty W_δ-subset of $CL(X)$ is of the 2nd category in itself.*

Proof. (1) We will construct a winning strategy σ for β in $Ch(CL(X))$. Denote $\Delta = \{(x,x) \in X : x \in \omega_1\}$, and put $\sigma(\emptyset) = (A_0, \mathbf{V}_0)$, where $A_0 = \{\omega_1\} \times \omega_1 \cup \{(x_0, y_0)\}$, and $\mathbf{V}_0 = (X \setminus \Delta)^+ \cap \{(x_0, y_0)\}^-$, where $x_0 > y_0$, and $(x_0, y_0) \notin \Delta$ is an isolated point of X. If $\mathbf{U}_0 = W_0^+ \cap \bigcap_{i \le k_0} W_{0,i}^- \in \mathcal{B}_V$ is α's first step, then $A_0 \in \mathbf{U}_0 \subseteq \mathbf{V}_0$. It follows that $\{\omega_1\} \times \omega_1 \subset W_0$, so we can find $x_1 > x_0$ such that $(x_1, x_0) \in W_0$ is isolated in X. Denote $y_1 = x_0$, $A_1 = A_0 \cup \{(x_1, y_1)\}$, $\mathbf{V}_1 = \mathbf{U}_0 \cap \{(x_1, y_1)\}^-$, and put $\sigma(\mathbf{U}_0) = (A_1, \mathbf{V}_1)$.

Assume that given $n \in \omega$ and $j \le n$, we have defined

$$(A_j, \mathbf{V}_j) = \sigma(\mathbf{U}_0, \ldots, \mathbf{U}_{j-1}) \text{ whenever } (\mathbf{U}_0, \ldots, \mathbf{U}_{j-1}) \in \mathcal{B}_V^j,$$

so that $\{\omega_1\} \times \omega_1 \cup \{(x_j, y_j)\} \subset A_j$ for some isolated point (x_j, y_j) of X such that

$$y_0 < x_0 = y_1 < x_1 = y_2 < \cdots < x_{n-1} = y_n < x_n.$$

Let $\mathbf{U}_n = W_n^+ \cap \bigcap_{i \le k_n} W_{n,i}^- \in \mathcal{B}_V$ be α's next choice, i.e. $A_n \in \mathbf{U}_n \subseteq \mathbf{V}_n$. It follows that $\{\omega_1\} \times \omega_1 \subset W_n$, so we can find $x_{n+1} > x_n$ such that $(x_{n+1}, x_n) \in W_n$ is isolated in X. Denote $y_{n+1} = x_n$, $A_{n+1} = A_n \cup \{(x_{n+1}, y_{n+1})\}$, $\mathbf{V}_{n+1} = \mathbf{U}_n \cap \{(x_{n+1}, y_{n+1})\}^-$, and put $\sigma(\mathbf{U}_0, \ldots, \mathbf{U}_n) = (A_{n+1}, \mathbf{V}_{n+1})$.

Claim 1. *σ is a winning strategy for β in $Ch(CL(X))$.*

Indeed, let β play according to σ, and assume there exists some $A \in \bigcap_n \mathbf{V}_n$. Then $A \in \mathbf{V}_0$, so $A \subset X \setminus \Delta$, moreover, $B = \{(x_n, y_n) : n \in \omega\} \subseteq A$. Since the sequences $(x_n), (y_n)$ converge to a common $x \in \omega_1$, then $(x, x) \in \overline{B} \subseteq A \subset X \setminus \Delta$, a contradiction.

(2) Let $\mathcal{M}$ be a nonempty W_δ-subset of $CL(X)$, and (G, T) a sieve of $\mathcal{M}$ in $CL(X)$ witnessing that $\mathcal{M}$ is a W_δ-set.

Claim 2. *$\exists M \in \mathcal{M}$ which is compact in X, i.e. there is some $\lambda < \omega_1$ such that $M \subseteq K(\lambda)$, where $K(\lambda) = [0, \lambda] \times (\omega_1 + 1) \cup (\omega_1 + 1) \times [0, \lambda]$.*

Indeed, take any $M_0 \in \mathcal{M}$. Let (t_n) be a branch in T so that $M_0 \in G(t_n)$ for each n, and without loss of generality, assume that each $G(t_n)$ is a τ_V-basic element, i.e. $G(t_n) = G_n^+ \cap \bigcap_{i \le m_n} U(z_{n,i})^- \in \mathcal{B}_V$, where $m_n \in \omega$, G_n is open in X, and $U(z_{n,i}) \subseteq G_n$ is a basic (compact) neighborhood of $z_{n,i} \in X$. Since $(G(t_n))_n$ is decreasing, it follows from Lemma 1.1 that, given n and $i \leq m_n$, there is $j \leq m_{n+1}$ such that $U(z_{n+1,j}) \subseteq U(z_{n,i})$, so we can assume that $m_{n+1} > m_n$, and that for all $i \leq m_n$, $U(z_{n+1,i}) \subseteq U(z_{n,i})$. Fix $n \in \omega$, and $i \leq m_n$. Then $M_0 \cap \bigcap_{p \geq n} U(z_{p,i})$ is a nonempty compact set, so we can choose $u_{n,i} \in M_0 \cap \bigcap_{p \geq n} U(z_{p,i})$. Then $M = \overline{\{u_{n,i} : n \in \omega, i \leq m_n\}}$ is clearly compact, moreover, $M \subseteq M_0 \subset G_n$ and $M \cap U(z_{n,i}) \neq \emptyset$ for each $n \in \omega, i \leq m_n$; thus, $M \in \bigcap_n G(t_n) \subseteq \mathcal{M}$.

It follows by Claim 2, that $\mathcal{M}_0 = \mathcal{M} \cap K(\lambda)^+$ is nonempty, and, as an open subspace of the W_δ-set $\mathcal{M}$, it is a W_δ-set. Furthermore, since $K(\lambda)$ is a clopen compact subspace of X, $CL(K(\lambda))$ is a clopen compact subspace of $CL(X)$. In summary, $\mathcal{M}_0$ is a W_δ-subset of the compact $CL(K(\lambda))$, so it is sieve-complete and, thus, of the 2nd category in itself. This implies that $\mathcal{M}$ is of the 2nd category in itself, since $\mathcal{M}_0$ is an open subspace of $\mathcal{M}$. □

References

[BLR] H. R. Bennett, D. J. Lutzer and G. M. Reed, *Domain representability and the Choquet game in Moore and BCO-spaces*, Topology Appl. 155 (2008), 445–458.

[CP] J. Cao and Z. Piotrowski, *Two variations of the Choquet game*, Kyungpook Math. J. 44 (2004), 495–504.

[CCN] J. Chaber, M. M. Čoban and K. Nagami, *On monotonic generalizations of Moore spaces, Čech complete spaces and p-spaces*, Fund. Math. 84 (1974), 107–119.

[Ch] G. Choquet, *Lectures on Analysis I.*, Benjamin, New York, 1969.

[De] G. Debs, *Espaces héréditairement de Baire*, Fund. Math. 129 (1988), 199–206.

[DM] F. G. Dorais and C. Mummert, *Stationary and convergent strategies in Choquet games*, Fund. Math. 209 (2010), 59–79.

[GT] F. Galvin and R. Telgársky, *Stationary strategies in topological games*, Topology Appl. 22 (1986), 51–69.

[Gr] G. Gruenhage, *Generalized metric spaces*, in Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984.

[HMC] R. C. Haworth and R. A. McCoy, *Baire spaces*, Dissertationes Math. 141 (1977), 1–77.

[Ke] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, New York, 1994.

[Kr] M. R. Krom, *Cartesian products of metric Baire spaces*, Proc. Amer. Math. Soc. 42 (1974), 588–594.

[Ma] K. Martin, *Topological games in domain theory*, Topology Appl. 129 (2003), 177–186.

[Mi] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. 71 (1951), 152–182.

[Ox] J. C. Oxtoby, *Cartesian products of Baire spaces*, Fund. Math. 49 (1961), 157–166.

[Po] E. Porada, *Jeu de Choquet*, Colloq. Math. 42 (1979), 345–353.

[Te] R. Telgársky, *Remarks on a game of Choquet*, Colloq. Math. 51 (1987), 365–372.

[WW] H. H. Wicke and J. M. Worrell, *On the open continuous images of paracompact Čech complate spaces*, Pacific J. Math. 37 (1971), 265–275.

[Zs] L. Zsilinszky, *On β-favorability of the strong Choquet game*, Colloq. Math. 125 (2011), 233–243.

Department of Mathematics and Computer Science, University of North Carolina at Pembroke, Pembroke, NC 28372, USA

E-mail address: laszlo@uncp.edu